1. Exercises from 3.3

Today we are going to work with surface paramaterizations as preparation for doing integration over surfaces.

PROBLEM 1. *Folland 3.3.1(b)*

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(u, v) = (au \cos v, bu \sin v, u)$ for $a, b > 0$.

- What happens if we fix $u = C$? Then the image of the map is curve (an ellipse) in $\mathbb{R}^3$ whose center is on the z-axis at a height $z = C$

$$f(C, v) = (Ca \cos v, Cb \sin v, C)$$

- If we fix $v = C$, then the image of the map is a line through the origin whose direction vector is $(a \cos C, b \sin C, 1)$
- These two facts together let us draw a sketch of the surface (actually draw the surface)
- Now let's write $f(u, v) = (x(u, v), y(u, v), z(u, v))$ and establish a functional relation $F(x, y, z) = 0$. (The trick is to exploit the trigonometric identity $\sin^2 v + \cos^2 v = 1$)
- $x(u, v)^2 + y(u, v)^2 = u^2(a^2 \cos^2 v + b^2 \sin^2 v)$, does not quite give us what we want
- $(bx)^2 + (ay)^2 = a^2b^2u^2 = (abz)^2$, so the functional relation describing the surface is

$$F(x, y, z) = b^2x^2 + a^2y^2 - a^2b^2z^2 = 0$$

- Now let's find the points where $\partial_u f$ and $\partial_v f$ are linearly independent. Our expectation from looking at our sketch is that the tangents to the surface will be linearly independent everywhere except the origin

$$\partial_u f = (a \cos v, b \sin v, 1)$$

$$\partial_v f = (-au \sin v, bu \cos v, 0)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos v & b \sin v & 1 \\ -au \sin v & bu \cos v & 0 \end{vmatrix} = (-bu \cos v, -au \sin v, abu)$$

- **Exercise**: Show that the above determinant is proportional to ∇F. Explain geometrically why this is exactly what you would expect.
- The coordinate vector fields are linearly dependent at the origin, which agrees with our sketch showing a singularity at the origin

PROBLEM 2. *Folland 3.3.2(a)*

- We need to find the coordinate tangent vectors at the surface at the point $(1, -2, 1)$, then take their cross product. The first step is to find which values of (u, v) give $f(u, v) = (1, -2, 1)$.
- $x = 1 = \exp(u - v) \Rightarrow u = v$
- $y = -2 = u - 3v = -2u \Rightarrow u = v = 1$
- Checking for consistency, $z = 1 = \frac{1}{2}(u^2 + v^2) = 1$.
- So we need to find the coordinate tangent vectors at the point $(u, v) = (1, 1)$.

$$\partial_u x(u, v)|_{(1,1)} = e^{u-v}|_{(1,1)} = 1$$

$$\partial_u y(u, v)|_{(1,1)} = 1$$

$$\partial_u z(u, v)|_{(1,1)} = u|_{(1,1)} = 1$$

$$\partial_v x(u, v)|_{(1,1)} = -e^{u-v}|_{(1,1)} = -1$$

$$\partial_v y(u, v)|_{(1,1)} = -3$$

$$\partial_v z(u,v)|_{(1,1)} = v|_{(1,1)} = 1$$

- So our two coordinate tangent vectors at the point $(u,v) = (1,1)$ are:

$$\partial_u f\Big|_{(1,1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \partial_v f\Big|_{(1,1)} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

- We can find the equation of the tangent plane by finding the normal vector and using the fact that we have a point $(1,-2,1)$ on the plane:

$$\vec{c} = \partial_u f\Big|_{(1,1)} \times \partial_v f\Big|_{(1,1)} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

$$4x - 2y - 2z = D \Rightarrow 4(1) - 2(-2) - 2(1) = 6 = D$$

So the equation of the tangent plane is:

$$4x - 2y - 2z = 6$$

PROBLEM 3. *(Folland 3.3.5(a)) Let S be the circle formed by intersecting the plane $x+z=1$ with the sphere $x^2+y^2+z^2=1$. Find a parametrization of the curve.*

- Draw picture
- The problem is much easier to solve in a coordinate system where we rotate first by $\pi/4$ around the y-axis. Define a new set of coordinates (x', y', z') such that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

- Nothing happens to the sphere when we rotate, but the plane $x+z=1$ becomes the plane:

$$\frac{1}{\sqrt{2}}(x'+z') + \frac{1}{\sqrt{2}}(-x'+z') = \sqrt{2}z' = 1 \Rightarrow z' = \frac{1}{\sqrt{2}}$$

- So we now want to parameterize the intersection of the plane $z' = 1/\sqrt{2}$ with the sphere $(x')^2 + (y')^2 + (z')^2 = 1$.
- Switching the spherical polar coordinates, we can see that any curve living in a plane of constant z' must have θ constant, and since $z' = \cos\theta = 1/\sqrt{2}$ we must have $\theta = \pi/4$.
- We can also see that $\phi \in [0, 2\pi]$, so simply let $\phi = t$.
- In polar coordinates, the equation of the curve is $(r, \phi, \theta) = (1, t, \pi/4)$, which we can easily convert to coordinates on $\mathbb{R}^3$:

$$\gamma(t) = \begin{pmatrix} r(t)\cos\phi(t)\sin\theta(t) \\ r(t)\sin\phi(t)\sin\theta(t) \\ r(t)\cos\theta(t) \end{pmatrix} = \begin{pmatrix} \frac{\cos t}{\sqrt{2}} \\ \frac{\sin t}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- Now we rotate back to our original coordinate system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{\cos t}{\sqrt{2}} \\ \frac{\sin t}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+\cos t}{2} \\ \frac{\sin t}{\sqrt{2}} \\ \frac{1-\cos t}{2} \end{pmatrix} = \begin{pmatrix} \cos^2(t/2) \\ \sin t/\sqrt{2} \\ \sin^2(t/2) \end{pmatrix}$$

- In retrospect, we could have just noticed from the beginning that $x+z=1$ naturally lends itselft to a parameterization such as $x(t) = \cos^2(t)$ and $z(t) = \sin^2(t)$.